

Relationship between quantum mechanics with and without monopoles

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Abstract

We show that the inclusion of the monopole field in the three- and five-dimensional spherically symmetric quantum mechanical systems, supplied by the addition of the special centrifugal term, does not yield any change in the radial wavefunction and in the functional dependence of the energy spectra on quantum numbers. The only change in the spectrum is the lift of the range of the total and azimuth quantum numbers. The changes in the angular part wavefunction are independent of the specific choice of the (central) potential. We also present the integrable model of the spherical oscillator which is different from the Higgs oscillator.

1 Introduction

During the last decades there was much activity in the study of the integrable quantum-mechanical systems specified by the presence of monopole-like field configurations. It was initiated by the pioneer works by Zwanziger [1] and McIntosh and Cisneros [2], where the analog of Coulomb problem with a Dirac monopole has been suggested, which inherits whole (nonlinear) symmetry algebra of the Coulomb system (MICZ-Kepler system). The similarity between MICZ-Kepler and Coulomb systems has quite transparent explanation in terms of four-dimensional space: these systems could be obtained from the four-dimensional oscillator by the reduction by $U(1)$ group action [3]. In the same way, one can construct the five-dimensional analog of the MICZ-Kepler problem, reducing the eight-dimensional oscillator by $SU(2)$ group action [4]. In this case, instead of Dirac monopole the $SU(2)$ Yang monopole appears in the system. The uniqueness of these reduction procedures insists on their close relation with the first and second Hopf maps (the detailed quantum-mechanical description of this correspondence could be found in [5]). Let us mention, that the existing analogs of these Coulomb-like systems on the curved spaces are also specified with the closed similarity of the systems with and without monopoles [6]. The actual observable difference between these systems results in the lift of the range of the total angular momentum, which in its turn leads to the degeneracy of the ground state.

In this paper we show, that the closed similarity between rotationally invariant three- /five- dimensional quantum-mechanical systems with and without Dirac/Yang monopole is the general peculiarity of these systems. We consider the quantum mechanics with central potential on the $d = 3, 5$ - dimensional spaces equipped with $so(d)$ -invariant metrics

$$ds^2 = g(r)d\mathbf{r}d\mathbf{r}, \quad r = |\mathbf{r}|, \quad \mathbf{r} = (x_1, \dots, x_d). \quad (1.1)$$

We shall show, that incorporation of the monopole supplying with the addition to the potential of the specific “centrifugal term”,

$$U(r) \rightarrow U(r) + \frac{\tilde{s}^2}{2g(r)r^2}, \quad (1.2)$$

yields the minor changes of the properties of the system (here $\tilde{s}^2 = \hbar^2 s^2$ with s is the monopole number for the $d = 3$ case, and $\tilde{s}^2 = \hbar^2 s(s + 1)$ with s is the isospin of the system for the $d = 5$ case). Namely, upon this modification, the radial wavefunction of the system, as well as the functional dependence of the energy spectrum on quantum numbers remain unchanged. The incorporation of the monopole affects the range of the $SO(d)$ orbital quantum number: it lifts the lowest admissible value of the orbital quantum number from 0 to s . The angular part of the wavefunction also changes, but this change is independent of the specific form of the central potential. In other words, *for any exactly solvable spherically symmetric three-/five-dimensional quantum-mechanical system without monopole we present its explicitly solved generalization with Dirac/Yang monopole*. Besides the obvious relation with quantum field theory, these systems could be useful in condensed matter. “Particle-Dirac monopole” configuration could be used for the description of the charge in the vicinity of the magnetic pole, while “particle-Yang monopole” configuration reduced to low dimension, could be used for the description of the systems with spin-orbit interaction. The curved spaces are also related with condensed matter: they correspond to the systems with effective non-constant masses. Let us mention, that incorporation of the monopole, not only provides the system with degenerate ground state, but also makes possible the dipole transitions preserving orbital quantum number [7]. Hence, incorporating the monopole in the existing quantum dot models, one can provide them by these specific peculiarities.

The paper is arranged as follows: In *Section 2* we consider the three-dimensional systems with Dirac monopole, in *Section 3* we consider the five dimensional systems with Yang monopole, and in *Section 4* we propose the model of the spherical oscillator which is different from the Higgs oscillator [10].

2 Dirac monopole

Let us consider the particle on the space equipped with a $so(3)$ -invariant conformal flat metric (1.1) moving in the Dirac monopole magnetic field $\mathbf{B} = \tilde{s}\mathbf{r}/|\mathbf{r}|^3$ (in the field) and in the $so(3)$ -invariant potential field $\tilde{s}^2/2r^2g + U(|\mathbf{r}|)$. Classical trajectories of this system are independent of monopole number \tilde{s} [8]. Let us show, that the same phenomenon takes place in the quantum-mechanical level. Corresponding quantum-mechanical system is given by the following Hamiltonian and scalar product

$$\hat{\mathcal{H}}_s = -\frac{\hbar^2}{2}\Delta_s + U(r), \quad (\Psi_1, \Psi_2) = \int \psi_1^* \psi_2 \sqrt{\det g} d^3x, \det g = \det g_{ij} \quad (2.1)$$

The generators of $SO(3)$ rotations are defined by the expressions

$$\mathbf{J} = \mathbf{r} \times \mathbf{p} - \tilde{s} \frac{\mathbf{r}}{|\mathbf{r}|}. \quad (2.2)$$

Here

$$\Delta_s = \frac{1}{2\det g^{1/2}} \hat{p}_i g^{ik} \det g^{1/2} \hat{p}_k, \quad \hat{p}_i = -i\hbar \frac{\partial}{\partial x_i} - \hbar s A_i, \quad s = \tilde{s}/\hbar \quad (2.3)$$

where A_i is vector potential of a Dirac monopole:

$$A_i = \frac{1}{r(r+x_3)}(-x_2, x_1, 0) : \nabla \times \mathbf{A} = \mathbf{r}/r^3.$$

Notice, that the momentum operator is not self-conjugated with respect to above scalar product. However, it could be easily regularized by the appropriate gauge transformation, preserving the Laplace operator

$$\hat{p}_i \rightarrow \hat{\tilde{p}}_i = \hat{p}_i - \frac{i\hbar}{4\det g} \frac{\partial \det g}{\partial x^i} : (\hat{\tilde{p}}_i \Psi_1, \Psi_2) = (\Psi_1, \hat{\tilde{p}}_i \Psi_2), \quad (2.4)$$

$$\hbar^2 \Delta_s = -\frac{1}{\det g^{1/4}} \hat{\tilde{p}}_i g^{ik} \det g^{1/2} \hat{\tilde{p}}_k \frac{1}{\det g^{1/4}} = -\frac{1}{\det g^{1/2}} \hat{p}_i g^{ik} \det g^{1/2} \hat{p}_k. \quad (2.5)$$

Now, let us specify our formulae for the metrics $g_{ij} = g(r)\delta_{ij}$, $\det g_{ij} = g^3$. Transiting to the spherical coordinates and taking into account the expression

$$\nabla \cdot \mathbf{A} = \frac{1}{\det g^{1/2}} \frac{\partial}{\partial x^i} (\det g^{1/2} g^{ik} A_k) = 0, \quad (2.6)$$

one could represent the Hamiltonian as follows

$$\hat{\mathcal{H}}_s = -\frac{\hbar^2}{2}\Delta_r + \frac{\hat{J}^2}{2gr^2} + U, \quad \Delta_r \equiv \frac{1}{g^{3/2}r^2} \frac{\partial}{\partial r} \left(g^{1/2}r^2 \frac{\partial}{\partial r} \right). \quad (2.7)$$

Here $\hat{\mathbf{J}}$ is the quantized angular momentum of the system, given by (2.2), so that

$$\hat{J}^2 = -\hbar^2 \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right] + \frac{2\hbar^2}{1 + \cos \theta} \left[s^2 - is \frac{\partial}{\partial \varphi} \right]. \quad (2.8)$$

The appropriate spectral problem reads

$$\hat{\mathcal{H}}_s \Psi_{E,j,m_j} = E \Psi_{E,j,m_j}, \quad \hat{J}^2 \Psi_{E,j,m_j} = \hbar^2 j(j+1) \Psi_{E,j,m_j}, \quad \hat{J}_z \Psi_{E,j,m_j} = \hbar m_j \Psi_{E,j,m_j} \quad (2.9)$$

The variables could be separated by the following choice of the wavefunction

$$\Psi_{n_r,j,m_j}(r, \theta, \phi) = \psi_{n_r,j,m_j}(r) e^{im_j \phi} d_{m_j s}^j(\theta), \quad (2.10)$$

where $d_{m_j s}^j$ is Wigner D -function. This substitution resolves the last two equations in (2.9), with quantization condition

$$j = |s|, |s| + 1, \dots, \quad m_j = -j, -j + 1, \dots, j - 1, j \quad s = 0, \pm 1/2, \pm 1 \dots \quad (2.11)$$

Then we get the following radial Schroedinger equation

$$\Delta_r \psi_{n_r,j,m_j} + \frac{2}{\hbar^2} \left[E_{n_r,j} - U(r) - \frac{j(j+1)}{g(r)r^2} \right] \psi_{n_r,j,m_j} = 0. \quad (2.12)$$

The radial quantum number n_r is defined from the above radial Schroedinger equation, and boundary conditions, and depends on the specific choice of potential $U(r)$. It is seen, that the radial Schroedinger equation is independent of the monopole number s . Hence, the spectrum of the Hamiltonian (2.1) remains unchanged after exclusion of the Dirac monopole. The only impact of the presence of Dirac monopole in the spectrum is in the range of definition of the orbital and azimuth quantum numbers j, m_j given by (2.11). The impact of the Dirac monopole in the wavefunction concerns spherical part, while radial wavefunction remains unchanged.

3 Yang monopole

Now let us consider the five-dimensional $SO(5)$ -invariant quantum mechanics with the $SU(2)$ Yang monopole. The Hamiltonian of this system is again given by the expression (2.1), where the momenta operators \hat{p}_i in (2.3) are replaced by the following ones

$$\hat{P}_i = \hat{p}_i - \hbar A_i^a \hat{T}_a, \quad a = 1, 2, 3, \quad [\hat{T}_a, \hat{T}_b] = \imath \varepsilon_{abc} \hat{T}_c. \quad (3.1)$$

The five-dimensional vector potential \mathbf{A}^a is defined by the expressions

$$\mathbf{A}^1 = \frac{1}{r(r+x_5)}(x_4, x_3, -x_2, -x_1, 0), \quad \mathbf{A}^2 = \frac{1}{r(r+x_5)}(-x_3, x_4, -x_1, -x_2, 0), \quad \mathbf{A}^3 = \frac{1}{r(r+x_5)}(x_2, -x_1, x_4, -x_3, 0) \quad (3.2)$$

and obey the equations

$$\mathbf{A}^a \cdot \mathbf{A}^b = \frac{1}{r^2} \frac{r-x_5}{r+x_5} \delta^{ab}, \quad \mathbf{A}^a \cdot \mathbf{r} = 0, \quad \nabla \cdot \mathbf{A}^a = 0 \quad (3.3)$$

This potential describes the Yang monopole with topological charge +1. We shall restrict ourselves by the detailed consideration of this case, since the transition to the system with the Yang anti-monopole (i.e. the Yang monopole with topological charge -1) is straightforward.

Let us pass to the five-dimensional Euler coordinates

$$x_5 = r \cos \theta, \quad x_1 + \imath x_2 = r \sin \theta \cos \frac{\beta}{2} e^{\imath \frac{\alpha+\gamma}{2}}, \quad x_3 + \imath x_4 = r \sin \theta \sin \frac{\beta}{2} e^{\imath \frac{\alpha-\gamma}{2}}. \quad (3.4)$$

In these coordinates the Hamiltonian looks as follows

$$\hat{H} = -\frac{1}{2} \hbar^2 \frac{1}{g^{5/2} r^4} \frac{\partial}{\partial r} \left(g^{3/2} r^4 \frac{\partial}{\partial r} \right) - \frac{\hbar^2 \hat{\Lambda}^2}{gr^2} + U, \quad \hat{\Lambda}^2 \equiv -\frac{1}{\sin^3 \theta} \frac{\partial}{\partial \theta} \left(\sin^3 \theta \frac{\partial}{\partial \theta} \right) + \frac{\hat{L}^2}{\sin^2(\theta/2)} + \frac{\hat{J}^2}{\cos^2(\theta/2)}. \quad (3.5)$$

Here \hat{L}_a and \hat{J}_a are the generators of $so(4) = so(3) \times so(3)$ algebra,

$$\hat{J}_a = \hat{L}_a + \hat{T}_a : \quad [\hat{L}_a, \hat{T}_b] = 0, \quad [\hat{L}_a, \hat{L}_b] = \imath \varepsilon_{abc} \hat{L}_c, \quad [\hat{J}_a, \hat{J}_b] = \imath \varepsilon_{abc} \hat{J}_c \quad (3.6)$$

and $\hat{\Lambda}^2$ defines the total $SO(5)$ momentum of the system. Explicitly the generators \hat{L}_a look as follows

$$\hat{L}_1 = \imath \left(\cos \alpha \cot \beta \frac{\partial}{\partial \alpha} + \sin \alpha \frac{\partial}{\partial \beta} - \frac{\cos \alpha}{\sin \beta} \frac{\partial}{\partial \gamma} \right), \quad \hat{L}_2 = \imath \left(\sin \alpha \cot \beta \frac{\partial}{\partial \alpha} - \cos \alpha \frac{\partial}{\partial \beta} - \frac{\sin \alpha}{\sin \beta} \frac{\partial}{\partial \gamma} \right), \quad \hat{L}_3 = -\imath \frac{\partial}{\partial \gamma}. \quad (3.7)$$

The dimensions of the quantum mechanics with and without Yang monopole are different ones, because of non-Abelian nature of the Yang monopole. Namely, in the absence of the Yang monopole the system is five-dimensional one, and its wavefunction depends on \mathbf{r} coordinates only. When we incorporate in the system the Yang monopole, we should take into account its internal space, the two-dimensional sphere S^2 (the maximal orbit of $SU(2)$ group). However, it is more convenient to consider the three-dimensional sphere S^3 instead of S^2 . For this purpose we define the generators \hat{T}_a in terms of the angles $\alpha_T, \beta_T, \gamma_T$, parameterizing three-dimensional sphere S^3 . The explicit expressions are given by (3.7), where \hat{L}_a are replaced by \hat{T}_a , and α, β, γ by the $\alpha_T, \beta_T, \gamma_T$. In that case one can consider the wavefunction depending on coordinates $(r, \theta, \alpha, \beta, \gamma, \alpha_T, \beta_T, \gamma_T)$, with the additional constraint imposed by

$$\hat{T}^2 \Psi(r, \theta, \alpha, \beta, \gamma, \alpha_T, \beta_T, \gamma_T) = s(s+1) \Psi(r, \theta, \alpha, \beta, \gamma, \alpha_T, \beta_T, \gamma_T), \quad (3.8)$$

where s is positive integer defining the isospin of the particle. So, after inclusion of the Yang monopole the initial five-dimensional system becomes six-dimensional one. Now, let us introduce the separation ansatz

$$\Psi(r, \theta, \alpha, \beta, \gamma, \alpha_T, \beta_T, \gamma_T) = R(r) Z(\theta) \Phi(\alpha, \beta, \gamma, \alpha_T, \beta_T, \gamma_T). \quad (3.9)$$

which resolves the following spectral problem

$$\mathcal{H} \Psi = \mathcal{E} \Psi, \quad \hat{\Lambda}^2 \Psi = \Lambda(\Lambda+3) \Psi, \quad (3.10)$$

$$\hat{L}^2 \Psi = L(L+1) \Psi, \quad \hat{T}^2 \Psi = s(s+1) \Psi, \quad \hat{J}^2 \Psi = J(J+1) \Psi, \quad \hat{J}_3 \Psi = M \Psi. \quad (3.11)$$

The functions Φ are the eigenfunctions of $\hat{L}^2, \hat{T}^2, \hat{J}^2, \hat{J}_3$ with the eigenvalues $L(L+1), s(s+1), J(J+1)$ and M respectively. Hence, Φ could be represented in the form

$$\Phi = \sum_{M=m+t} (JM|L, m'; s, t') D_{mm'}^L(\alpha, \beta, \gamma) D_{tt'}^s(\alpha_T, \beta_T, \gamma_T) \quad (3.12)$$

where $(JM|L, m'; s, t')$ are the Clebsh-Gordan coefficients and $D_{mm'}^L$ and $D_{tt'}^s$ are the Wigner functions. These quantum numbers have the following range of definition

$$L = 0, 1, \dots; \quad J = |L - s|, |L - s| + 1, \dots, |L + s| - 1, |L + s|; \quad M = -J, -J + 1, \dots, J - 1, J. \quad (3.13)$$

The function $Z(\theta)$ is the eigenfunction of the $SO(5)$ momentum operator:

$$\hat{\Lambda}^2 Z(\theta) = \Lambda(\Lambda + 3)Z(\theta) : \quad Z_{\Lambda L J} = C_{\Lambda L J} (1 - \cos \theta)^L (1 + \cos \theta)^J P_{\Lambda - L - J}^{(2L+1, 2J+1)}(\cos \theta), \quad (3.14)$$

where $C_{\Lambda L J}$ is normalization constant,

$$C_{\Lambda L J} = \sqrt{\frac{(2\Lambda + 3)(\Lambda - J - L)! \Gamma(\Lambda + J + L + 3)}{2^{2J+2L+3} \Gamma(\Lambda + J - L + 2) \Gamma(\Lambda - J + L + 2)}}, \quad (3.15)$$

and $P_{\Lambda - L - J}^{(2L+1, 2J+1)}$ are Jacobi polynomials. Hence, we get the following quantization condition

$$\Lambda = n_\theta + L + J, \quad n_\theta = 0, 1, \dots \quad (3.16)$$

Substituting (3.9) into (3.5) and taking into account appropriate eigenvalues we get the equation for the radial wavefunction

$$\frac{1}{g^{5/2} r^4} \frac{d}{dr} \left(g^{3/2} r^4 \frac{dR}{dr} \right) - \frac{\Lambda(\Lambda + 3)}{gr^2} R + \frac{2}{\hbar^2} (E - U) R = 0. \quad (3.17)$$

It is seen that the radial Schroedinger equation is independent of the isospin s , so that the impact of the Yang monopole in the spectrum results in the change of the range of quantum number Λ (and of J and M as well). The impact of the Yang monopole in the wavefunction concerns spherical part and is independent of the specific form of the potential, while radial wavefunction remains unchanged.

Consideration of the system with Yang anti-monopole (Yang monopole with -1 topological charge) is completely similar to the above one. The respective formulae can be obtained by the given ones by the replacement

$$J \rightarrow L, \quad L \rightarrow J. \quad (3.18)$$

Hence, we get that the consequences of the inclusion of the Yang (anti-)monopole in the five-dimensional $SO(5)$ symmetric system supplemented with the appropriate change of potential given by (1.2), is completely similar to the inclusion of Dirac monopole in the three-dimensional $SO(3)$ -symmetric system.

4 Spherical oscillators

In previous Sections we proposed the explicit expressions for the wavefunctions and spectra of the generalizations of three-/five- dimensional spherically symmetric exactly solvable quantum mechanical systems (without external gauge fields). Clearly, our construction includes, as particular cases, the MICZ-Kepler systems on the three- and five- dimensional Euclidean spaces, spheres and hyperboloids (see second and third references in [6], as well as Ref. [9], where the MICZ-Kepler system on the any-dimensional spheres and hyperboloid has been constructed). For example, in conformal flat coordinates the metric of sphere and two-sheet hyperboloid look as follows

$$g d\mathbf{r}^2 = \frac{4r_0^2 d\mathbf{r}^2}{(1 + \varepsilon r^2)^2}, \quad \varepsilon = \pm 1 \quad (4.1)$$

where $\varepsilon = 1$ corresponds to the sphere, and $\varepsilon = -1$ corresponds to the (two-sheet) hyperboloid, and r_0 is the radius of the sphere (hyperboloid). In these coordinates the potential of the Higgs oscillator reads

$$U_{Higgs} = \frac{\omega^2 r_0^2}{2} \frac{4r^2}{(1 - \varepsilon r^2)^2}. \quad (4.2)$$

The energy spectrum of the Higgs oscillator spectrum of this system is given by the expression [11]

$$E_{n_r, \Lambda} = \frac{1}{2r_0^2} \left[\left(2\sqrt{\omega^2 r_0^4 + \frac{1}{4}} - 1 \right) \left(2n_r + \Lambda + \frac{d}{2} \right) + \varepsilon \left(2n_r + \Lambda + \frac{\varepsilon + 1}{2} \right) \left(2n_r + \Lambda + d + \frac{\varepsilon - 1}{2} \right) \right]. \quad (4.3)$$

Here Λ denotes the orbital quantum number, reducing to $\Lambda = j$ for $d = 3$. The radial quantum number n_r takes the values $n_r = 0, 1, \dots$ for $\varepsilon = 1$, and $n_r = 0, 1, \dots, [\sqrt{\omega^2 r_0^4 + \frac{1}{4}} - d/2 - \Lambda/2]/2$ for $\varepsilon = -1$. Incorporation of the monopole yields the change of the range of Λ from $\Lambda = 0, 1, \dots$ to $\Lambda = s, s+1, \dots$

Let us conclude this Section proposing the alternative model of the spherical oscillator. It was initially suggested as a model of oscillator on the complex projective space \mathbb{CP}^N and quaternionic projective space \mathbb{HP}^N [12] respecting the inclusion of constant magnetic field (on \mathbb{CP}^N) and BPST instanton field (on \mathbb{HP}^N) [13]. For the $N = 1$, i.e. $\mathbb{CP}^1 = S^2$ and $\mathbb{HP}^1 = S^4$ the corresponding potentials are defined by the expression

$$V(\mathbf{r}) = 2\omega^2 r_0^2 r^2 = 2\omega^2 r_0^2 \frac{1 - \cos \chi}{1 + \cos \chi}, \quad (4.4)$$

where χ is the spherical coordinate of the respective sphere, $x_d = r_0 \cos \chi$.

Surprisingly, this potential defines the integrable generalization of the oscillator in the arbitrary-dimensional spheres too. Indeed, with the formulae for the Schroedinger equation of the spherically-symmetric system on the d -dimensional sphere at hand [11], one could immediately get the energy spectrum and the wavefunctions for the system with potential. Namely, the spectrum of the system is defined by the expression

$$E_{n_r, \Lambda} = \frac{1}{2r_0^2} \left[(2n_r + \Lambda + 1)(2n_r + \Lambda + d) + (2\nu - 1)(2n_r + \Lambda + \frac{d}{2}) \right], \quad \nu^2 = (\Lambda + \frac{d-2}{2})^2 + 16\omega^2 r_0^4 \quad (4.5)$$

where $n_r = 0, 1, \dots$ is radial quantum number, and $\Lambda = 0, 1, \dots$ is the $SO(d+1)$ orbital quantum number. The radial wavefunction looks as follows

$$R_{n_r, \Lambda \nu}^D(\chi) = C_{n_r, \Lambda \nu} (\sin \chi)^\Lambda (\cos \chi)^{\nu - d/2 + 1} P_{n_r}^{(\Lambda + d/2 - 1, \nu)}(\cos \chi) \quad (4.6)$$

where $P_n^{(a,b)}$ is Jacobi polinomial and $C_{n_r, \Lambda \nu}$ is normalization constant

$$C_{n_r, \Lambda \nu} = \frac{1}{2^{(d-1)/2}} \sqrt{\frac{(2n_r + \Lambda + \nu + d/2)n_r! \Gamma(n_r + \Lambda + \nu + d/2)}{r_0^d \Gamma(n_r + \Lambda + d/2) \Gamma(n_r + \nu + 1)}}. \quad (4.7)$$

One can expect, that this oscillator model will respect the inclusion of the $SO(d)$ instanton field not only in $d = 2, 4$, but in any d too. It is also clear, that the same potential will define the integrable system on the d -dimensional two-sheet hyperboloid.

Conclusion

We have shown, that the incorporation of the Dirac/Yang monopole in the spherically symmetric exactly solvable three-/five-dimensional quantum mechanical system (without external gauge fields), supplemented by the change of the potential given by Eq. (1.2), yields the exactly solvable generalization of the initial system. Moreover, we have shown, that the radial wavefunction remains unchanged upon given modification, as well as the functional dependence of the energy spectrum on radial orbital and azimuth quantum numbers. Hence, having at hand the exact solution of the quantum-mechanical system without monopole, we can immediately present the exact solutions of the respective system with monopole. We suppose, that the similar correspondence takes place not only in $d = 3, 5$ dimensional spherically symmetric systems, but for the arbitrary d -dimensional $SO(d)$ -symmetric system, upon inclusion of $SO(d-1)$ monopole field. Also, we present some exactly solvable model of the oscillator on the d -dimensional sphere: besides the $SO(d-1)$ monopole field, this model would, presumably, respect the inclusion of $SO(d)$ instanton field too. Such a monopole and instanton solutions are given in [14].

The similar analysis of the systems with axial symmetry is more complicated but more important (the simplest system of this sort, specified with the presence of Dirac monopole was proposed in [15]). Indeed, presented way of the incorporation of the monopole in the spherically symmetric systems does not yield essential change in the system's properties. However, their interaction with the external fields (without spherical symmetry) could yield qualitatively different effects, as it was observed in the study of the Stark effect in the MICZ-Kepler system [16].

Acknowledgments. The work has been supported in part by the grants NFSAT-CRDF ARP1-3228-YE-04 and INTAS-05-7928.

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